3. The time-dependent Green's function or propagator

A free particle of mass m in one-dim.

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

the propagator

$$K(x_2, x_1; t) \equiv \langle x_2 | e^{-i\hat{H}t/\hbar} | x_1 \rangle = \langle x_2 | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t} | x_1 \rangle$$

completeness $\int_{-\infty}^{\infty} dp \, |p\rangle\langle p|$

$$= \int dp \, \langle x_2 | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t} | p \rangle \langle p | x_1 \rangle$$

$$= \int dp \, \langle x_2 | p \rangle \langle p | x_1 \rangle \, e^{-\frac{i}{\hbar}\frac{p^2}{2m}t}$$

$$= \int dp \, \frac{1}{2\pi\hbar} \cdot \exp\left[-\frac{i}{\hbar}\left\{\frac{p^2}{2m}t - p(x_2 - x_1)\right\}\right]$$

$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \, \exp\left[-\frac{i}{\hbar}\frac{t}{2m}\left(p - \frac{m(x_2 - x_1)}{t}\right)^2 + \frac{i}{\hbar}\frac{m(x_2 - x_1)^2}{2t}\right]$$

$$= \frac{1}{2\pi\hbar} \cdot \exp\left[\frac{i}{\hbar}\frac{m(x_2 - x_1)^2}{2t}\right] \sqrt{\frac{2m\hbar\pi}{it}}$$

$$= \sqrt{\frac{m}{2\pi\hbar i t}} \exp\left[\frac{i}{\hbar}\frac{m}{2t}(x_2 - x_1)^2\right]$$

as shown in class.

Application : Wave function at $t = 0$

$$\Psi(x; 0) = \frac{\alpha^{1/4}}{\pi^{1/4}} \cdot e^{-\frac{\alpha}{2}(x - x_0)^2} e^{ip_0 x/\hbar}$$

$$\rightarrow \Psi(x; t) = \int dx' \, K(x, x'; t) \, \Psi(x'; 0)$$

$$\Psi(x;t) = \int_{-\infty}^{\infty} dx' \sqrt{\frac{m}{2\pi\hbar i t}} \exp\left[\frac{im}{\hbar 2t}(x-x')^2\right] \frac{\alpha^{1/4}}{\pi^{1/4}} \cdot e^{-\frac{\alpha}{2}(x'-x_0)^2 + \frac{ip_0}{\hbar}}$$

$$= \sqrt{\frac{m}{2\pi\hbar i t}} \frac{\alpha^{1/4}}{\pi^{1/4}} \exp\left[\frac{im}{\hbar 2t}x^2 - \frac{\alpha}{2}x_0^2\right] \times$$

$$\times \int dx' \exp\left[\frac{im}{\hbar 2t}x'^2 - \frac{im}{\hbar t}xx' - \frac{\alpha}{2}x'^2 + \alpha x' x_0 + \frac{ip_0}{\hbar}x'\right]$$

$$= f(x) \times \int_{-\infty}^{\infty} dx' \exp\left[-\left(\frac{\alpha}{2} - \frac{im}{2\hbar t}\right)x'^2 + \left(\frac{ip_0}{\hbar} + \alpha x_0 - \frac{im}{\hbar t}x\right)x'\right]$$

From Integral Table

$$\int_{-\infty}^{\infty} dx \exp[-ax^2+bx] = \frac{\pi^{1/2}}{a^{1/2}} e^{b^2/4a}$$

$$= f(x)\frac{\pi^{1/2}}{\left(\frac{\alpha}{2}-\frac{im}{2\hbar t}\right)^{1/2}} \cdot \exp\left[\frac{\left(\frac{ip_0}{\hbar}+\alpha x_0 - \frac{im}{\hbar t}x\right)^2}{2(\alpha - im/\hbar t)}\right]$$

$$= \sqrt{\frac{m}{\hbar i t}} \cdot \frac{\alpha^{1/4}}{\pi^{1/4}} \left(\alpha - \frac{im}{\hbar t}\right)^{-1/2} \times$$

$$\exp\left\{\frac{im}{2\hbar t}x^2 - \frac{\alpha}{2}x_0^2 + \frac{1}{2}\left(\frac{ip_0}{\hbar}+\alpha x_0 - \frac{im}{\hbar t}x\right)^2\left(\alpha - \frac{im}{\hbar t}\right)^{-1}\right\}$$

Time dependent Schrödinger equation

$$i\hbar \frac{\partial\Psi}{\partial t} \stackrel{?}{=} \frac{\hat{p}^2}{2m}\Psi(x;t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x;t)$$

$$\frac{\partial}{\partial x}\Psi(x;t) = \text{const}\,\Big\{\frac{im}{\hbar t}x + \left(\frac{ip_0}{\hbar}+\alpha x_0 - \frac{im}{\hbar t}x\right)\left(-\frac{im}{\hbar t}\right)\Big(\alpha - \frac{im}{\hbar t}$$

· exponetral term

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi = -\frac{\hbar^2}{2m}\text{const}\,\Big\{\underline{\frac{im}{\hbar t} + \left(-\frac{im}{\hbar t}\right)\left(-\frac{im}{\hbar t}\right)\left(\alpha - \frac{im}{\hbar t}\right)^{-1}}$$

$$-\frac{m^2}{\hbar^2 t^2}x^2 + \left(\frac{ip_0}{\hbar}+\alpha x_0 - \frac{im}{\hbar t}x\right)^2\left(\frac{-m^2}{\hbar^2 t^2}\right)\left(\frac{\hbar t}{\alpha\hbar t - im}\right)^2$$

$$\langle \Psi(t) | x^2 | \Psi(t) \rangle = (\text{const}')^* (\text{const}')$$

$$\int_{-\infty}^{\infty} dx \; \exp\left[\frac{m}{\alpha\hbar t + im} \left\{ \frac{-\alpha i}{2} x^2 + \left(\frac{P_0}{\hbar} + i\alpha x_0 \right) x \right\} \cdot x^2 \cdot \right.$$

$$\cdot \exp\left[\frac{m}{\alpha\hbar t - im} \left\{ \frac{\alpha i}{2} x^2 + \left(\frac{P_0}{\hbar} - i\alpha x_0 \right) x \right\} \right.$$

$$= \int_{-\infty}^{\infty} dx \; x^2 \cdot \exp\left[\frac{1}{(\alpha\hbar t)^2 + m^2} \left\{ -m^2 x^2 + \left(\frac{2P_0\alpha\hbar t}{\hbar} + \alpha x_0 m \right) x \right\} \right]$$

Integral Table

$$\int_{-\infty}^{\infty} dx \; x^2 e^{-Ax^2 + Bx} = e^{\frac{B^2}{4A}} \left(\int_{-\infty}^{\infty} dx \; x^2 e^{-Ax^2} dx + \frac{B^2}{4A^2} \int_{-\infty}^{\infty} e^{-Ax^2} dx \right)$$

$$\int_{-\infty}^{\infty} dx \; x \, e^{-Ax^2 + Bx} = e^{\frac{B^2}{4A}} \frac{B}{2A} \int_{-\infty}^{\infty} e^{-Ax^2} dX$$

$$\Delta X(t) = \left[e^{\frac{B^2}{4A}} \int x^2 e^{-Ax^2} dx + e^{\frac{B^2}{4A}} \frac{B^2}{4A^2} \int e^{-Ax^2} dx \right.$$

$$\left. - e^{\frac{B^2}{2A}} \frac{B^2}{4A^2} \left(\int e^{-Ax^2} dx \right)^2 \right]$$

$$= e^{\frac{B^2}{4A}} \cdot \frac{1}{2} \left(\frac{\pi}{A} \right)^{1/2} \frac{1}{A} + e^{\frac{B^2}{4A}} \frac{B^2}{4A^2} \cdot \left(\frac{\pi}{A} \right)^{1/2} - e^{\frac{B^2}{2A}} \frac{B^2}{4A^2} \cdot \left(\frac{\pi}{A} \right)$$

$$= e^{\frac{B^2}{4A}} \left(\frac{\pi}{A} \right)^{1/2} \left\{ \frac{1}{2A} + \frac{B^2}{4A^2} - \frac{B^2}{4A^2} \left(\frac{\pi}{A} \right)^{1/2} e^{\frac{B^2}{4A}} \right\}$$

where $A = \dfrac{-m^2}{(\alpha\hbar t)^2 + m^2}$,

$$B = \frac{1}{(\alpha\hbar t)^2 + m^2} \left(\frac{2P_0\alpha\hbar t}{\hbar} + \alpha x_0 m \right)$$

//

where $\text{const} = \left(\dfrac{m}{\alpha\hbar i t + m} \right)^{1/2} \dfrac{\alpha^{1/4}}{\pi^{1/4}}$

$$+ 2\left(\frac{m}{\hbar t}\right)^2 \left(\alpha - \frac{im}{\hbar t}\right)^{-1} x \left(\frac{ip_0}{\hbar} + \alpha x_0 - \frac{im}{\hbar t} x \right) \Big\}$$ · exponetral term

$$i\hbar \frac{\partial}{\partial t} \Psi(x;t) = i\hbar \frac{m^{1/2}\alpha^{1/4}}{\pi^{1/4}} \cdot \left(-\frac{1}{2}\right) (\alpha\hbar i t + m)^{-\frac{3}{2}} \cdot (\alpha\hbar i)$$ exponential

$$+ i\hbar \cdot \text{const} \cdot \Big(-\frac{im}{2\hbar} x^2 t^{-2} + \left(\frac{ip_0}{\hbar} + \alpha x_0 - \frac{im}{\hbar t} x \right)\left(\frac{im}{\hbar} x t^{-2} \right)\left(\alpha - \frac{im}{\hbar t} \right)^{-1}$$

$$+ \frac{1}{2}\left(\frac{ip_0}{\hbar} + \alpha x_0 - \frac{im}{\hbar t} x \right)^2 \frac{-\hbar i m}{(\alpha\hbar t - im)^2} \Big)$$ · exponential

You can easily match those two equations term by term "

"Uncertainty in position"

$$\Delta x(t) = \left[\langle \Psi(t) | x^2 | \Psi(t) \rangle - \langle \Psi(t) | x | \Psi(t) \rangle^2 \right]^{1/2}$$

$$\Psi(x;t) = \text{const} \times \exp\Big[\frac{1}{\alpha\hbar t - im} \Big(\frac{\alpha i m}{2} x^2 + \frac{\alpha i m}{2} x_0^2 +$$

$$+ \left(\frac{p_0 m}{\hbar} - i m \alpha x_0 \right) x + i p_0 \alpha t x_0 - \frac{p_0^2 t}{2\hbar} \Big)$$

$$= \text{const}' \exp\Big[\frac{m}{\alpha\hbar t - im} \Big\{ \frac{\alpha i}{2} x^2 + \left(\frac{p_0}{\hbar} - i\alpha x_0 \right) x \Big\} \Big]$$

where

$$\text{const}' = \left(\frac{m}{\alpha\hbar i t + m} \right)^{1/2} \frac{\alpha^{1/4}}{\pi^{1/4}} \exp\Big[\frac{1}{\alpha\hbar t - im} \Big\{ \frac{\alpha i m}{2} x_0^2 + i p_0 \alpha t x_0 - \frac{p_0^2 t}{2\hbar} \Big\} \Big]$$