Problem 3

Part (a)

$$\begin{aligned}\langle H\rangle(t) &= \langle\psi(t)|H|\psi(t)\rangle \\ &= \langle\psi(0)|U^\dagger(t)HU(t)|\psi(0)\rangle \\ &= \langle\psi(0)|e^{iHt/\hbar}He^{-iHt/\hbar}|\psi(0)\rangle\end{aligned}$$

because the propagator commutes with the Hamiltonian

$$\begin{aligned}\langle H\rangle(t) &= \langle\psi(0)|He^{iHt/\hbar}e^{-iHt/\hbar}|\psi(0)\rangle \\ &= \langle\psi(0)|H|\psi(0)\rangle \qquad \text{using the unitarity of propagators} \\ &= \langle H\rangle(0)\end{aligned}$$

Therefore, the expectation value of energy is conserved for a time independent Hamiltonian.

Part (b)

In the Heisenberg picture, $H_H = H$ using the unitarity of the propagator

$$\begin{aligned}i\hbar\frac{dH_H}{dt} &= [H_H, H_H] + i\hbar\frac{\partial H_H}{\partial t} \qquad (1) \\ &= 0\end{aligned}$$

The right hand side is zero because the Hamiltonian commutes with itself, and the partial derivative with respect to time is zero.

Part (c)

If the Hamiltonian is explicitly time dependent, the partial derivative in (1) will not be zero, though the commutator will remain unchanged. So,

$$\frac{dH_H}{dt} = \frac{\partial H_H}{\partial t}$$

So, the energy will not be conserved in this case.

Energy is conserved if and only if the Hamiltonian is time independent.