Time Evol Problem 1

From the Heisenberg eqs of motion,

$$i\hbar \frac{d}{dt} \hat{x}^H(t) = -[\hat{H}, \hat{x}]^H \quad \text{and} \quad i\hbar \frac{d}{dt} \hat{p}^H(t) = -[\hat{H}, \hat{p}]^H.$$

Since $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$,

$$[\hat{H}, \hat{x}] = \left[\frac{\hat{p}^2}{2m}, \hat{x}\right] \quad \text{and}$$

$$[\hat{p}^2, \hat{x}] = \hat{p}[\hat{p}, \hat{x}] + [\hat{p}, \hat{x}]\hat{p} = -2i\hbar \hat{p} \quad \text{so} \quad \left[\frac{\hat{p}^2}{2m}, \hat{x}\right] = -\frac{i\hbar}{m}\hat{p}.$$

$$i\hbar \frac{d}{dt}\langle x\rangle_t = i\hbar \frac{d}{dt}\langle \psi(t)|\hat{x}|\psi(t)\rangle = i\hbar \frac{d}{dt}\langle \psi(0)|\hat{x}^H(t)|\psi(0)\rangle$$

$$= -\langle \psi(0)|[\hat{H}, \hat{x}]^H|\psi(0)\rangle = +\frac{i\hbar}{m}\langle \psi(0)|\hat{p}^H(t)|\psi(0)\rangle = +\frac{i\hbar}{m}\langle p\rangle_t. \Rightarrow$$

$$m\frac{d}{dt}\langle x\rangle_t = \langle p\rangle_t.$$

Next, $[\hat{H}, \hat{p}]\phi(x) = [\hat{V}, \hat{p}]\phi(x) = \hat{V}\left(-i\hbar\frac{\partial}{\partial x}\right)\phi(x) - \left(-i\hbar\frac{\partial}{\partial x}\right)\hat{V}\phi(x)$

$$= -i\hbar \hat{V}\phi'(x) + i\hbar V'\phi(x) + i\hbar \hat{V}\phi'(x) = i\hbar V'\phi(x) \Rightarrow$$

$$[\hat{H}, \hat{p}] = i\hbar V' = i\hbar \frac{\partial V}{\partial x}.$$

$$i\hbar \frac{d}{dt}\langle p\rangle_t = i\hbar \frac{d}{dt}\langle \psi(0)|\hat{p}^H(t)|\psi(0)\rangle = -\langle \psi(0)|[\hat{H}, \hat{p}]^H|\psi(0)\rangle$$

$$= -i\hbar\langle \psi(0)|\left(\frac{\partial V}{\partial x}\right)^H|\psi(0)\rangle = -i\hbar\left\langle \frac{\partial V}{\partial x}\right\rangle_t \Rightarrow$$

$$\frac{d}{dt}\langle p\rangle_t = -\left\langle \frac{\partial V}{\partial x}\right\rangle_t$$