Problem 5

Initially, the electron is in the $1s$ state. Let the electric field be $H^1 = EtZ$

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\begin{aligned}
d_{210}(t) &= -\frac{i}{\hbar}\int_0^t \langle 210|Z|100\rangle Ete^{i\omega t}\,dt \\
\langle 210|Z|100\rangle &= \iiint \frac{1}{\sqrt{32\pi a_0^3}}\frac{r}{a_0}e^{-\frac{r}{2a_0}}\cos\theta(rcos\theta)\frac{1}{\sqrt{\pi a_0^3}}e^{-\frac{r}{a_0}}r^2\sin\theta\,drd\theta d\phi \\
&= \frac{1}{4\sqrt{2}\pi a_0^4}\int_0^{2\pi} d\phi\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta\int_0^{\infty} r^4 e^{-\frac{3r}{2a_0}}\,dr \\
&= \frac{1}{4\sqrt{2}\pi a_0^4}\times 2\pi\times\left(\frac{2}{3}\right)\times\int_0^{\infty} r^4 e^{-\frac{3r}{2a_0}}\,dr \\
\text{Let } \alpha &= \frac{3}{2a_0} \\
\int_0^{\infty} r^4 e^{-\alpha r}\,dr &= \int_0^{\infty}\frac{\partial^4}{\partial\alpha^4}e^{-\alpha r}\,dr \\
&= \frac{d^4}{d\alpha^4}\int_0^{\infty} e^{-\alpha r}\,dr \\
&= \frac{d^4}{d\alpha^4}\frac{1}{\alpha} \\
&= \frac{24}{\alpha^5} \\
&= 24\times\left(\frac{2a_0}{3}\right)^5 \\
&= \frac{256a_0^5}{81} \\
\therefore \langle 210|Z|100\rangle &= \frac{1}{4\sqrt{2}\pi a_0^4}\times 2\pi\times\left(\frac{2}{3}\right)\times\frac{256a_0^5}{81} \\
&= \frac{128\sqrt{2}}{243}a_0
\end{aligned}
$$

$$
\begin{aligned}
d_{210}(t) &= -\frac{i}{\hbar}\int_0^t \frac{128\sqrt{2}}{243} a_0 E t e^{i\omega t}\, dt \\
&= -\frac{128\sqrt{2}}{243\hbar} a_0 E \int_0^t i t e^{i\omega t}\, dt \\
&= -\frac{128\sqrt{2}}{243\hbar} a_0 E \int_0^t \frac{\partial}{\partial\omega} e^{i\omega t}\, dt \\
&= -\frac{128\sqrt{2}}{243\hbar} a_0 E \frac{d}{d\omega} \int_0^t e^{i\omega t}\, dt \\
&= -\frac{128\sqrt{2}}{243\hbar} a_0 E \frac{d}{d\omega} \left(\frac{e^{i\omega t} - 1}{i\omega} \right) \\
&= -\frac{128\sqrt{2}}{243\hbar} a_0 E \left(\frac{i t e^{i\omega t}}{i\omega} - \frac{e^{i\omega t} - 1}{i\omega^2} \right) \\
&= -\frac{128\sqrt{2}}{243\hbar} a_0 E \left(\frac{(i + t\omega) e^{i\omega t} - i}{\omega^2} \right) \\
P_{2\leftarrow 1}(t) &= |d_{210}(t)|^2 \\
&= \left(\frac{32768}{59049}\right) \left(\frac{a_0}{\hbar}\right)^2 \frac{E^2}{\omega^4} \big((i + t\omega)e^{i\omega t} - i\big)\big((-i + t\omega)e^{-i\omega t} + i\big) \\
&= \left(\frac{32768}{59049}\right) \left(\frac{a_0}{\hbar}\right)^2 \frac{E^2}{\omega^4} \big(2 + t^2\omega^2 - 2\cos(\omega t) - 2t\omega\sin(\omega t)\big) \\
\omega &= \frac{Ry}{\hbar}\left(1 - \frac{1}{4}\right) \\
&= \frac{3Ry}{4\hbar}
\end{aligned}
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