Problem 4

General Derivation

The eigenstates, $|n^0\rangle$ of H^0 form a complete basis. So, let us expand our wavefunction in terms of these.

$$H(t) = H^0 + H^1(t)$$
$$|\psi(t)\rangle = \sum_n c_n(t)|n^0\rangle$$
$$c_n(t) = d_n(t)e^{-iE_n^0 t/\hbar}$$
$$\Rightarrow |\psi(t)\rangle = \sum_n d_n(t)e^{-iE_n^0 t/\hbar}|n^0\rangle$$

d_n is not independent of time because of H^1

$$i\hbar|\dot{\psi}(t)\rangle = H|\psi(t)\rangle$$
$$\Rightarrow i\hbar\sum_n \left(\dot{d}_n(t) - \cancel{\frac{iE_n^0}{\hbar}d_n(t)}\right)e^{-iE_n^0 t/\hbar}|n^0\rangle = \cancel{\sum_n d_n(t)e^{-iE_n^0 t/\hbar}H^0|n^0\rangle} + H^1|\psi(t)\rangle$$
$$\sum_n \left(i\hbar\dot{d}_n(t) - H^1(t)d_n(t)\right)e^{-iE_n^0 t/\hbar}|n^0\rangle = 0$$

Taking the overlap with $\langle m^0(t)|$

$$i\hbar\dot{d}_m(t) = \sum_n d_n(t)\langle m^0|e^{iE_m^0 t/\hbar}H^1(t)e^{-iE_n^0 t/\hbar}|n^0\rangle$$
$$= \sum_n d_n(t)\langle m^0|H^1(t)|n^0\rangle e^{i\omega_{mn}t} \qquad \text{where } \omega_{mn} = \frac{E_m^0 - E_n^0}{\hbar}$$

Let us start from the state $|i^0\rangle$. So, $d_m(0) = \delta_{mi}$ Zeroth order solution is $\dot{d}_m(t) = 0 \Rightarrow d_m(t) = \delta_{mi}$ First order solution

$$\dot{d}_m(t) = \frac{-i}{\hbar}\langle m^0|H^1|i^0\rangle e^{i\omega_{mi}t}$$
$$d_m(t) = d_m(0) - \frac{i}{\hbar}\int_0^t \langle m^0|H^1|i^0\rangle e^{i\omega_{mi}t}\,dt$$
$$d_m(t) = \delta_{mi} - \frac{i}{\hbar}\int_0^t \langle m^0|H^1|i^0\rangle e^{i\omega_{mi}t}\,dt$$

Applying to perturbed harmonic oscillator

$$\begin{aligned} H^1(t) &= \lambda X \\ &= \lambda\sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right) \end{aligned}$$

We have prepared the system in the ground state of the harmonic oscillator

$$\begin{aligned} d_n &= \delta_{n0} - \frac{i}{\hbar}\int_0^t \lambda\sqrt{\frac{\hbar}{2m\omega}}\langle n^0|a + a^\dagger|0\rangle e^{in\omega t}\, dt \\ a|0\rangle &= 0 \\ a^\dagger|0\rangle &= |1\rangle \\ d_n &= \delta_{n0} - \frac{i}{\hbar}\lambda\sqrt{\frac{\hbar}{2m\omega}}\int_0^t \langle n^0|1\rangle e^{in\omega t}\, dt \\ d_n &= \delta_{n0} - \delta_{n1}\lambda\frac{i}{\sqrt{2m\omega\hbar}}\int_0^t e^{in\omega t}\, dt \\ &= \delta_{n0} - \delta_{n1}\lambda\frac{1}{\sqrt{2m\omega\hbar}}\frac{e^{in\omega t} - 1}{n\omega} \end{aligned}$$

If $m = 1$

$$\begin{aligned} d_1(t) &= -\frac{\lambda}{\sqrt{2m\omega\hbar}}\frac{e^{i\omega t} - 1}{\omega} \\ P_{1\leftarrow 0}(t) &= |d_1(t)|^2 \\ &= \frac{\lambda^2}{2m\omega\hbar}\frac{\left(2 - e^{i\omega t} - e^{-i\omega t}\right)}{\omega^2} \\ &= \frac{\lambda^2}{m\omega^3\hbar}\left(1 - \cos(\omega t)\right) \\ &= \frac{2\lambda^2}{m\omega^3\hbar}\sin^2\left(\frac{\omega t}{2}\right) \end{aligned}$$

If $m = 2$, $d_2(t) = P_{2\leftarrow 0}(t) = 0$

Because of the symmetry of the harmonic oscillator, the wavefunctions alternate between even and odd with increasing quantum number. We can justify the transition probabilities by looking at the nature of the integrand $\psi_n^*(x)x\psi_0(x)$.