Problem 11

$$H = \begin{pmatrix} 0 & -\hbar\Omega \\ -\hbar\Omega & 0 \end{pmatrix}$$

The energies of the eigenstates are $\pm\hbar\Omega$. So the partition function

$$Z = \sum_{states} e^{-\beta E(state)} = e^{-\hbar\beta\Omega} + e^{\hbar\beta\Omega} = 2\cosh\left(\hbar\beta\Omega\right)$$

Normalized boltzmann operator in the eigenbasis of the TLS would be

$$\rho = \frac{1}{Z}\begin{pmatrix} e^{-\hbar\beta\Omega} & 0 \\ 0 & e^{\hbar\beta\Omega} \end{pmatrix} = \frac{1}{2\cosh(\hbar\beta\Omega)}\begin{pmatrix} e^{-\hbar\beta\Omega} & 0 \\ 0 & e^{\hbar\beta\Omega} \end{pmatrix}$$

The right- and left- localized states have a general form

$$\frac{|+\rangle \pm |-\rangle}{\sqrt{2}}$$

where $|\pm\rangle$ are the eigenstates with energies $\pm\hbar\Omega$

$$\begin{aligned} \text{Population of the localized states} &= \frac{\langle+| \pm \langle-|}{\sqrt{2}}\frac{1}{Z}\left(e^{-\hbar\beta\Omega}|+\rangle\langle+| + e^{\hbar\beta\Omega}|-\rangle\langle-|\right)\frac{|+\rangle \pm |-\rangle}{\sqrt{2}} \\ &= \frac{1}{2Z}\left(e^{-\hbar\beta\Omega}\langle+|+\rangle\langle+|+\rangle + e^{\hbar\beta\Omega}\langle-|-\rangle\langle-|-\rangle\right) \\ &= \frac{Z}{2Z} \\ &= \frac{1}{2} \end{aligned}$$

So, the populations of the right- and left- localized eigenstates are both 0.5.