Problem 10

Assuming $|L\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|R\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

1 Part (a)

$$\begin{aligned}
\rho &= \frac{1}{2}\left(|R\rangle\langle R| + |L\rangle\langle L|\right) + \frac{1}{4}\left(|R\rangle\langle L| + |L\rangle\langle R|\right) \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \\
Tr[\rho] &= \frac{1}{2} + \frac{1}{2} \\
&= 1
\end{aligned}$$

2 Part (b)

In its eigenbasis, a matrix is the diagonal matrix of its eigenvalues.

$$\begin{aligned}
\rho &= \begin{pmatrix} 0.5 & 0.25 \\ 0.25 & 0.5 \end{pmatrix} \\
det\begin{pmatrix} 0.5-\lambda & 0.25 \\ 0.25 & 0.5-\lambda \end{pmatrix} &= 0 \\
(0.5-\lambda)^2 - 0.25^2 &= 0 \\
\Rightarrow (0.5-\lambda-0.25)(0.5-\lambda+0.25) &= 0 \\
\Rightarrow \lambda &= 0.25; 0.75 \\
\therefore \rho &= \begin{pmatrix} 0.25 & 0 \\ 0 & 0.75 \end{pmatrix} \\
Tr[\rho] &= 0.25 + 0.75 \\
&= 1
\end{aligned}$$

3 Part (c)

$$\rho = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.75 \end{pmatrix}$$
$$\rho^2 = \begin{pmatrix} 0.0625 & 0 \\ 0 & 0.5625 \end{pmatrix}$$
$$Tr[\rho^2] = 0.0625 + 0.5625$$
$$= 0.625 \neq 1$$

So, this is a mixed ensemble.

4 Part (d)

$$S = -k_B \sum_n P_n \log(P_n)$$
$$= -k_B \left(0.25 \log(0.25) + 0.75 \log(0.75)\right)$$
$$= 0.56233 k_B$$